1. Mental Math (No Calculators Allowed)

Example:

Question 1:

Question 2:

Question 3:

Question 4:

Question 5:

Question 6:

Question 7:

Question 8:

Question 9:

Question 10:

2. Applications of Algebra: Cell Phone Charges (Calculators Allowed)

The table below lists charges for different types of cell phone uses. Each text message costs 3¢, each local call costs 5¢ per minute, and each long distance call costs 10¢ per minute. In the questions below, please specify all answers in dollars where appropriate.

Type of Use	Charge
Text	3¢ per text
Local Call	5¢ per min
Long Distance Call	10¢ per min

Problem 2.1. Determine how much it costs to send 7 text messages and talk for 1 hour if:

(A) (1 point) All the calls are local.

$

(B) (1 point) 75% of the calling time is local.

$

(C) (1 point) You spend four times as long on local calls as on long distance.

$

Problem 2.2. (2 points) You send 12 texts and use 110 minutes for a total cost of $8.11. Of these 110 minutes, how many were for long distance calls?

min

Problem 2.3. Due to a change in software, the phone company has decided to charge 8¢ for each local text and 13¢ for each long distance text.

(A) (1 point) You plan to send 17 local texts and speak for 22 minutes locally. How much does this cost?

$

(B) (2 points) How many local texts can you send if you have $9.18 and you use twice as many minutes on local calls as the number of local texts you send?

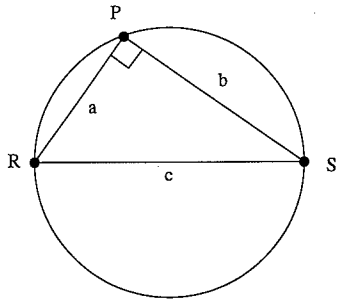
texts

(C) (2 points) You send twice as many local texts as long distance texts and speak for five times as long locally as you do long distance minutes. If you send twice as many texts as you use minutes, how many local minutes will you use for $25.67?

min

3. Geometry and Measurement (Calculators Allowed)

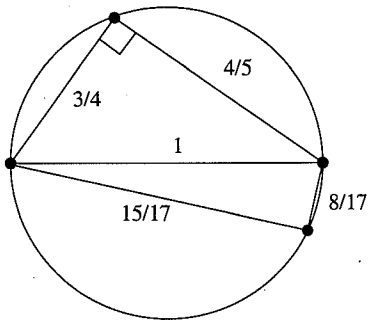
Circles have a very special property. Suppose R and S are points at opposite ends of a diameter, and P is any other point along the arc. When we draw a triangle connecting these points, the lines $\overline{RP}$ and $\overline{SP}$ will be perpendicular. (This means that the angle at P will always be 90 degrees.)

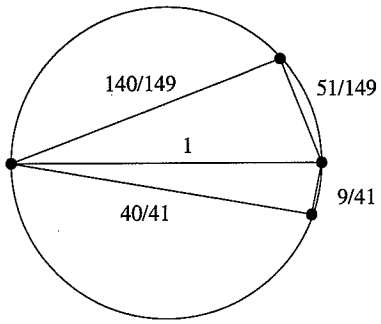


Because of this special property, the side lengths of the triangle satisfy

$$\boxed{a^2 + b^2 = c^2}$$

Here are some examples when the diameter is 1:





Notice that the triangles in the first circle satisfy

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \qquad \text{and} \qquad \left(\frac{15}{17}\right)^2 + \left(\frac{8}{17}\right)^2 = \frac{225}{289} + \frac{64}{289} = 1$$

Notice that if $a = \frac{A}{C}$, $b = \frac{B}{C}$ and $c = 1$, $a^2 + b^2 = c^2$ is the same as $\boxed{A^2 + B^2 = C^2}$.

Example: If $a = \frac{140}{149}$, $b = \frac{\square}{149}$ and $c = 1$, what is $\square$?

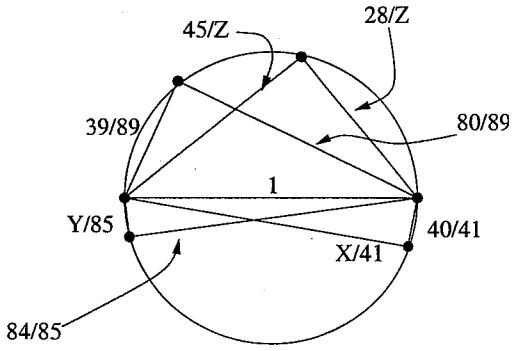
Solution. By the above comment, we must have that $140^2 + \square^2 = 149^2$. Plugging this into our calculator it means that

$$19600 + \square^2 = 22201$$
$$\square^2 = 22201 - 19600 = 2601$$

So $\square = \sqrt{2601} = 51$. Notice, by the way, that this is the same as the upper triangle in the second example circle above.) $\square$

In the problems below, keep in mind that the drawings may not be to scale.

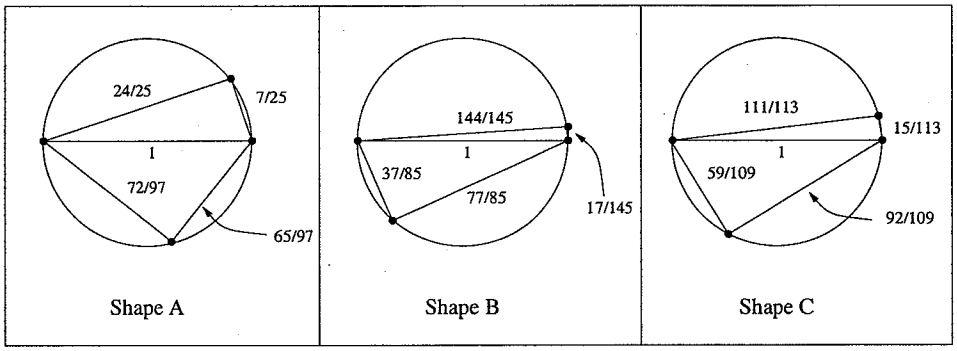
Problem 3.1. The following shape is a circle with diameter 1.



Find X, Y and Z. (Round you answers to the nearest whole number.)

$X =$ $Y =$ $Z =$

Problem 3.2. This special property about circles gives us a way of determining whether a given shape isn't a circle or if it might be a circle.



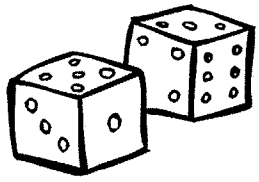
Is it possible that Shape A is a circle? (yes or no?)

Is it possible that Shape B is a circle? (yes or no?)

Is it possible that Shape C is a circle? (yes or no?)

4. Problem Solving: Probability (No Calculators Allowed)

A standard die has six sides. Each has an equal chance of being rolled.



Problem 4.1. Suppose you roll one die once.

(A) (1 point) What is the probability that you roll a 2?

(B) (1 point) What is the probability that you roll an even number?

(C) (1 point) What is the probability that you roll a multiple of 3?

Problem 4.2. Now suppose that you roll one die and then roll it again.

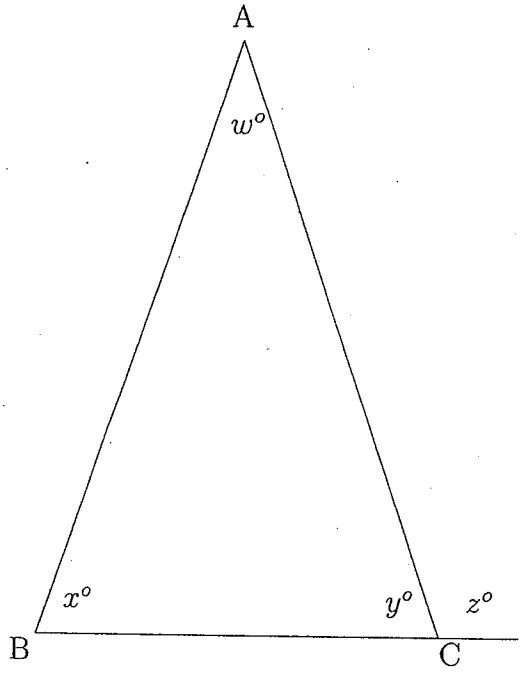
(A) (2 points) What is the probability that you roll a 1 the first time and then a 5 the second time?

(B) (2 points) What is the probability that you roll two numbers that add up to 3?

(C) (3 points) What is the probability that you roll two numbers that add up to 8?

5. Isosceles Triangles

Problem 5.1. (13 points) In the picture below, we have an isosceles triangle with $\overline{AB} = \overline{AC}$



(A) (1 point) Find an angle that equals the angle y^o.

(B) (2 points) What is the sum $y^o + z^o$?

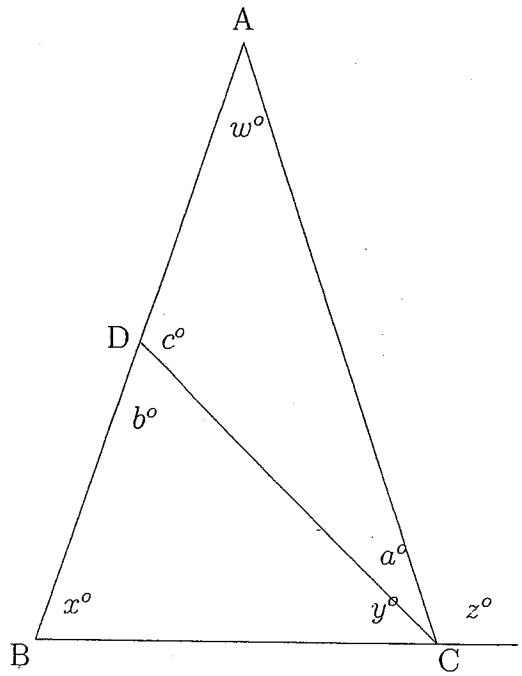
(C) (2 points) If $w = 68^o$, evaluate the angle x^o.

(D) (2 points) If $w = 10^o$, evaluate the angle z^o.

(E) (3 points) If $z = 120^o$, evaluate the angles x^o, y^o, w^o.

(F) (3 points) If z^o is three times as big as w^o, evaluate x^o.

Problem 5.2. (25 points) In the picture below is an isosceles triangle with $AB = AC$.



Questions

(A) (2 points) Evaluate $y^o + z^o + a^o + b^o + c^o$

(B) (1 point) Find an angle that equals $y^o + a^o$

(C) (2 points) If $x = b$ and $a = w$ find two lines in the shape that have the same length as line $\overline{BC}$

(D) (3 points) Evaluate the expression $z^o - x^o - w^o$.

(E) (5 points) If $y^o = w^o$ and $x = 72^o$, evaluate the angle a.

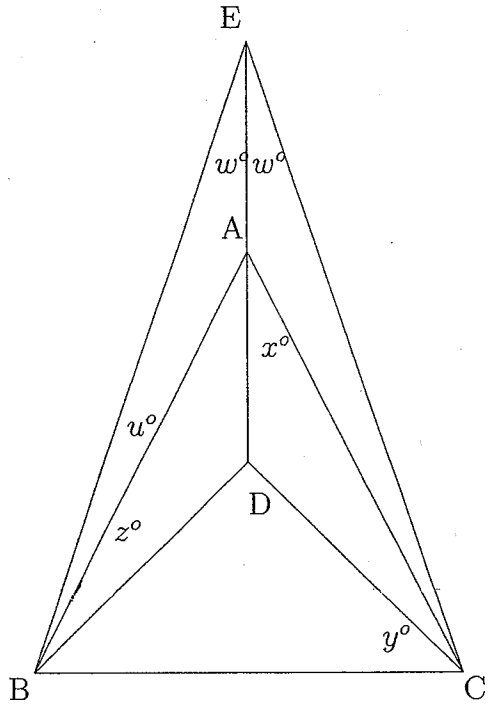
For the second part of this problem, we assume also that $BC = CD = DA$

(F) (2 points) How many isosceles triangles are now in the shape?

(G) (10 points) Find the angle w^o

Problem 5.3. (12 points)

In the picture below, $AB = AC$ and $DB = DC$ and $x^o = z^o$,



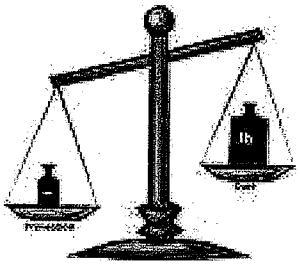
Questions

(A) (6 points) Evaluate the angle y^o.

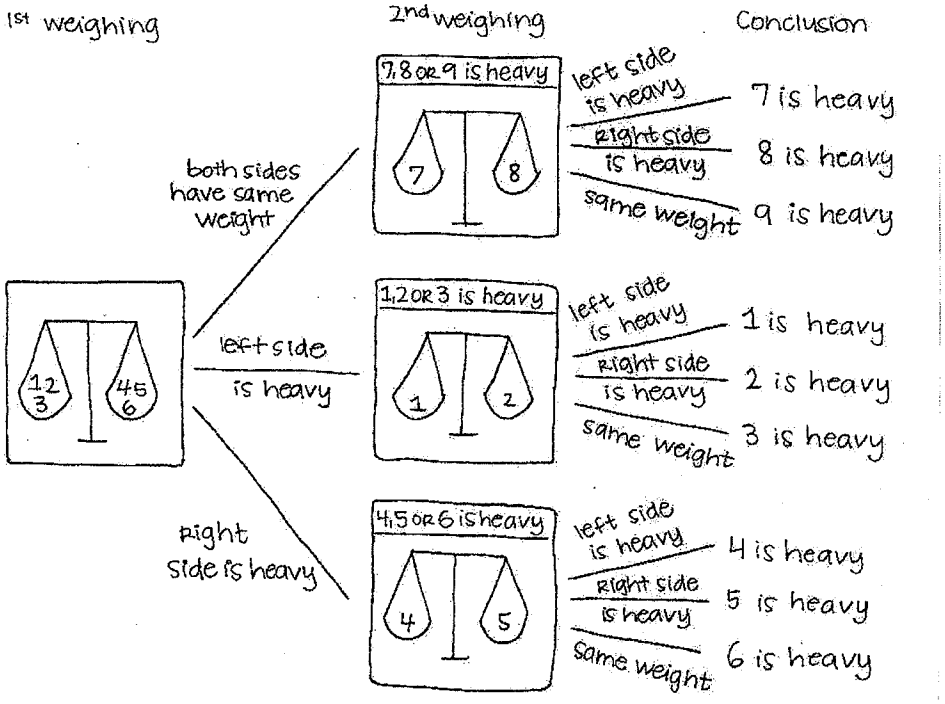
(B) (6 points) If you also know that $EB = EC$ and that u^o is half as big as z^o, find the angle w^o.

6. Team problem: Weighing Marbles

Example: You are given nine marbles, one of which is heavier than the other eight. Using only an old-fashioned scale (as in the picture below), devise a procedure by which you could determine which marble is heavier. How many weighings are necessary?



Solution. Number the marbles from 1 through 9, and divide then in three groups of three. Then proceed as illustrated in this flow chart:



To guarantee that you identify which is heaviest, you need to do two weighings. □

Note: For all of the problems below, you are allowed to put as many or as few marbles as you would like on either side of the scale and count it as a weighing. You cannot, however, repeatedly add one marble at a time thinking it's one weighing.

Problem 6.1. In each of the parts of this question, you are given a certain number of marbles, and in each case one of them is heavier than the others. Your task is to determine the minimum number of weighings needed to guarantee that you will find the heavy one.

(A) (3 points) If you are given three marbles, how many weighings do you need?

(B) (3 points) If you are given four marbles, how many weighings do you need then?

(C) (3 points) What about if you are given 27 marbles?

(D) (3 points) What if you are given ten marbles?

(E) (3 points) Lastly, how many weighings do you need if you are given eight marbles?

Problem 6.2. The game is now changed: you are allowed to use the scale only a certain number of times. This time your task is to determine what is the maximum number of marbles that you could be given, and be certain that you could determine which one of them is heavier than all the other ones.

(A) (3 points) If you were only allowed to use the scale once, what is the maximum number of marbles you could be given?

(B) (3 points) What if you can use the scale at most twice?

(C) (3 points) What about if you could use the scale at most three times, what is the maximum number of marbles you could be given then?

Problem 6.3. For this problem, we will repeat what has already been done in Problem 6.1, but now for numbers of the form

$$\underbrace{3 \times 3 \times \cdots \times 3}_{k\text{-times}}$$

For example,

$$3, \quad 9 = 3 \times 3, \quad 27 = 3 \times 3 \times 3, \quad 81 = 3 \times 3 \times 3 \times 3.$$

(A) (3 points) If you have $3 \times 3 \times 3 \times 3 = 81$ marbles, how many weighings do you need to determine for sure which is heavier?

(B) (3 points) What if you have $3 \times 3 \times 3 \times 3 \times 3 = 243$ marbles?

(C) (3 points) What if you have $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$ marbles?

(D) (5 points) In general, if you have $\underbrace{3 \times 3 \times \cdots \times 3}_{k\text{-times}}$ many marbles, how many weighings are needed to determine which is heaviest?

Problem 6.4. For this question, assume that we are continuing with the same set up, where you have a certain number of marbles, one of which is heavier than the other ones.

(A) (4 points) How many weighings will you need to determine the heavy one if you are given 242 marbles?

(B) (4 points) What if you are given 728 marbles?

(C) (4 points) What if you are given 730 marbles?

The question below is only going to be used as a tie-breaker. Solve it last, when you are done with all of the other team problems. Partial credit may be given, so do not hesitate to submit a partial solution.

This time around, suppose that you are given three marbles, one of which is different than the other two, but you do not know whether it is lighter or heavier. In other words, two of the marbles are the same weight, and the other one is either lighter or heavier than the other two. If you can only use the scale twice, draw a flow chart similar to the one in the example that shows how you would determine for sure which marble is different from the other two.